NONSTEADY CONVECTIVE HEAT EXCHANGE IN PRISMATIC PIPES OF TRIANGULAR CROSS SECTION

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The Kantorovich method and the method of characteristic curves are used successively to solve the nonsteady problem of convective heat exchange during the laminar flow of a viscous incompressible liquid in prismatic pipes having cross sections in the form of isosceles triangles, and an approximate analytical expression is obtained for the temperature field in the liquid.

The solution of nonsteady problems of convective heat exchange in pipes of "nonclassical" cross section by analytical methods is a very complicated task. The use of the method of the Laplace double integral transformation with the Bubnov–Galerkin method to solve such problems is hindered by the achievement of the transition back to the region of the inverse transforms [1]. We will obtain an approximate analytical solution to the nonsteady problem of heat exchange in the laminar hydrodynamically stabilized flow of a liquid inside a pipe having a cross section in the form of an isosceles triangle with a constant temperature T_0 of the liquid at the entrance, a constant distribution of the temperature T_0 at the initial time, and a constant temperature $T_w \neq T_0$ of the inner surface of the channel wall. The remaining assumptions are standard [2].

We formulate the boundary-value problem

$$\frac{\partial T}{\partial t} + \omega(x, y)\frac{\partial T}{\partial z} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{1}$$

$$(t > 0,\ z > 0,\ 0 < y < h_m,\ -y\,\mathrm{tg}\,\beta < x < y\,\mathrm{tg}\,\beta),$$

$$\begin{cases} T(x, y, z, 0) = T_0, & (2)\\ T(x, y, 0, t) = T_0, & (3)\\ T(x, y, z, t)|_S = T_w. & (4)\end{cases}$$

Here $T = T(x, y, z, t)$, t is the time, the z axis is directed along the axis of the stream, the x and y axes are located in the plane of the channel cross section, a is the coefficient of thermal diffusivity, and $\omega = \omega(x, y)$ is the velocity profile of the liquid flow in the pipe [2]:

$$\omega(x, y) = 3\omega_{\mathrm{av}}\frac{(B+2)(x^2 - y^2\,\mathrm{tg}^2\beta)}{(B-2)\,\mathrm{tg}^2\beta\, h_m^2}\left[\left(\frac{y}{h_m}\right)^{B-2} - 1\right],$$

where $B = \sqrt{4 + {}^5/_2[(1/\tan^2\beta) - 1]}$. The index S denotes the lateral surface of the pipe. We introduce the dimensionless quantities

$$\Theta = \frac{T - T_0}{T_w - T_0},\quad Z = \frac{1}{\mathrm{Pe}}\cdot\frac{z}{h_m},$$

$$\mathrm{Fo} = \frac{at}{h_m^2},\ X = \frac{x}{h_m},\ Y = \frac{y}{h_m},\ \mathrm{Pe} = \frac{\omega_{\mathrm{av}} h_m}{a}.$$

In dimensionless form the boundary-value problem is formulated as follows:

$$\frac{\partial\Theta}{\partial\mathrm{Fo}} + 3AW(X, Y)\frac{\partial\Theta}{\partial Z} = \frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Y^2}, \tag{5}$$

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$$\begin{cases} \Theta(X, Y, Z, 0) = 0, & (6) \\ \Theta(X, Y, 0, \mathrm{Fo}) = 0, & (7) \\ \Theta(X, Y, Z, \mathrm{Fo})/S_1 = 1, & (8) \end{cases}$$

where

$$A = \frac{B+2}{(B-2)\,\mathrm{tg}^2\beta}; \; S_1: \{Z > 0,\; 0 \leqslant Y \leqslant 1,\; X = \pm Y \,\mathrm{tg}\,\beta\},$$

$$W(X, Y) = [X^2 - Y^2 \mathrm{tg}^2\beta][Y^{B-2} - 1].$$

To solve the problem (5)-(8) we successively use the Kantorovich method [3] and the method of characteristic curves [4]. We represent the approximate solution of the problem in the form

$$\Theta_n(X, Y, Z, \mathrm{Fo}) = 1 + \sum_{k=1}^{n} a_k(Z, \mathrm{Fo})\,\Psi_k(X, Y), \tag{9}$$

where $\Psi_k(X, Y)$ is the system of coordinate functions ($k = 1, 2, \ldots, n$). The eigenfunctions of the Sturm–Liouville steady-state problem [2] are taken as the coordinate functions.

The method of R-functions [5] can be used in the selection of the coordinate functions in the case of more complicated cross-sectional profiles of prismatic pipes.

Using the Kantorovich method, for the determination of the unknown coefficient-functions $a_k(Z, \mathrm{Fo})$ we obtain a system of first-order linear differential equations in partial derivatives:

$$\sum_{k=1}^{n} A_{km}\frac{\partial a_k}{\partial \mathrm{Fo}} + B_{km}\frac{\partial a_k}{\partial Z} = \sum_{k=1}^{n} a_k D_{km}, \tag{10}$$

$$\begin{cases} A_{km} = A_{mk} = \iint\limits_D \Psi_k \Psi_m dD, \\ B_{km} = A_{mk} = 3A \iint\limits_D W \Psi_k \Psi_m dD, \\ D_{km} = D_{mk} = \iint\limits_D \Delta\Psi_k \Psi_m dD \end{cases}$$

with initial conditions of the form

$$\begin{cases} \sum\limits_{k=1}^{n} a_k(Z, 0) A_{km} = F_m, \\ \sum\limits_{k=1}^{n} a_k(0, \mathrm{Fo}) A_{km} = F_m, \end{cases} \tag{11}$$

where $F_m = -\iint\limits_D \Psi_m(X, Y)\,dD$, D is a region of the pipe cross section, Δ is the Laplace operator, and $m = 1, 2, \ldots, n$. From the conditions (6)-(8) and (11) it is seen that $a_k(Z, 0) = a_k(0, \mathrm{Fo}) = a_k(0, 0)$, $k = 1, 2, \ldots, n$.

We use the method of characteristic curves to solve the system (10), (11).

Let us consider the case when n = 2.

We have

$$\Psi_1(X, Y) = (247 - 1110X^2 + 45.5Y)\,W(X, Y),$$
$$\Psi_2(X, Y) = (152 - 11340X^2 + 31.6Y)\,W(X, Y).$$

In the second approximation (n = 2) the dimensionless temperature is obtained in the form

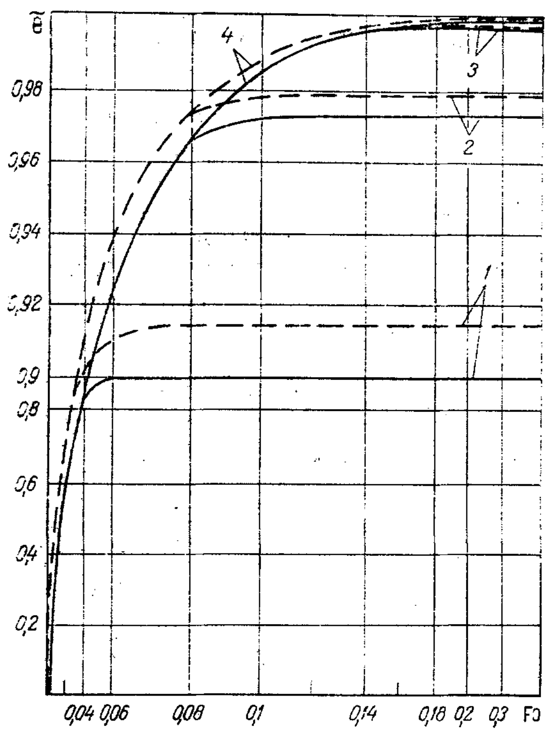

Fig. 1. Variation of average-mass temperature of liquid in a pipe with a cross section in the form of an isosceles triangle (β = π/5, solid lines; β = π/5.5, dashed lines). Respective values of Z: 1) 0.09; 2) 0.15; 3) 0.25; 4) 5.

$$\Theta(X, Y, Z, \mathrm{Fo}) = 1 + \sum_{k=1}^{2} \begin{cases} a_k^1(\mathrm{Fo})\,\Psi_k(X, Y), & Z > \mu_1\mathrm{Fo}, \\ a_k^2(Z, \mathrm{Fo})\,\Psi_k(X, Y), & \mu_2\mathrm{Fo} < Z < \mu_1\mathrm{Fo}, \\ a_k^3(Z)\,\Psi_k(X, Y), & Z < \mu_2\mathrm{Fo}. \end{cases} \tag{12}$$

The following expressions are obtained for the determination of the coefficients $a_k^1(\mathrm{Fo})$ and $a_k^3(Z)$ (k = 1, 2) in Eq. (12):

$$a_1^1(\mathrm{Fo}) = R\exp(v\mathrm{Fo}) - G\left[\frac{v^2+vl+g}{(v-p)(v-q)}\exp(v\,\mathrm{Fo}) + \frac{p^2+pl+g}{(p-v)(p-q)}\exp(p\,\mathrm{Fo}) + \frac{q^2+ql+g}{(q-v)(q-p)}\exp(q\,\mathrm{Fo})\right],$$

$$a_2^1(\mathrm{Fo}) = \frac{1}{p-q}\left[(Mp+N)\exp(p\,\mathrm{Fo}) - (Mq+N)\exp(q\,\mathrm{Fo})\right],$$

$$a_1^3(Z) = R\exp(uZ) - G\left[\frac{u^2+ub+d}{(u-f)(u-h)}\exp(uZ) + \frac{h^2+hb+d}{(h-f)(h-u)}\exp(hZ) + \frac{f^2+fb+d}{(f-u)(f-h)}\exp(fZ)\right],$$

$$a_2^3(Z) = \frac{1}{f-h}\left[(Mf+N_1)\exp(fZ) - (Mh+N_1)\exp(hZ)\right],$$

where

$$R = \frac{r_2}{\alpha_2},\quad G = \frac{r_2\alpha_1 - r_1\alpha_2}{\alpha_2(\alpha_1-\alpha_2)},\quad l = \frac{cr_2\beta_1 + \gamma_2 r_2\alpha_1 - cr_1\beta_2 - \gamma_2 r_1\alpha_2}{c(r_2\alpha_1 - r_1\alpha_2)},$$

$$g = \frac{\gamma_2(r_2\beta_1 - r_1\beta_2)}{c(r_2\alpha_1 - r_1\alpha_2)},\quad v = -\frac{\beta_2}{\alpha_2},\quad M = \frac{r_2\alpha_1 - r_1\alpha_2}{c(\alpha_1-\alpha_2)},$$

$$N=\frac{r_2\beta_1-r_1\beta_2}{c(\alpha_1-\alpha_2)},\quad p=\frac{-m+\sqrt{m^2-4n}}{2},\quad q=\frac{-m-\sqrt{m^2-4n}}{2},$$

$$m=\frac{c(\beta_1-\beta_2)+\gamma_2\alpha_1-\gamma_1\alpha_2}{c(\alpha_1-\alpha_2)},\quad n=\frac{\gamma_2\beta_1-\gamma_1\beta_2}{c(\alpha_1-\alpha_2)},$$

$$c=\begin{vmatrix} B_{12} & A_{12}\\ B_{22} & A_{22}\end{vmatrix},$$

$$\alpha_k=\mu_k\begin{vmatrix} A_{11} & A_{12}\\ A_{12} & A_{22}\end{vmatrix}+\begin{vmatrix} B_{12} & A_{11}\\ B_{22} & A_{21}\end{vmatrix}\quad k=1,\ 2,$$

$$\gamma_k=\mu_k\begin{vmatrix} A_{12} & D_{12}\\ A_{22} & D_{22}\end{vmatrix}+\begin{vmatrix} B_{22} & D_{22}\\ B_{12} & D_{21}\end{vmatrix}\quad k=1,\ 2,$$

$$\beta_k=\mu_k\begin{vmatrix} A_{12} & D_{11}\\ A_{22} & D_{12}\end{vmatrix}+\begin{vmatrix} B_{22} & D_{12}\\ B_{12} & D_{11}\end{vmatrix}\quad k=1,\ 2,$$

$$r_k=a_1(0,\ 0)\alpha_k+ca_2(0,\ 0),\ k=1,\ 2,$$

μ_k are the roots of the equation

$$\begin{vmatrix} B_{11}-\mu A_{11} & B_{12}-\mu A_{12}\\ B_{12}-\mu A_{12} & B_{22}-\mu A_{22}\end{vmatrix}=0,$$

with $\mu_1 > \mu_2$,

$$u=-\frac{\beta_2}{\mu_2\alpha_2},\quad b=\frac{c(r_2\mu_2\beta_1-r_1\mu_1\beta_2)+\gamma_2(r_2\mu_1\alpha_1-r_1\mu_1\alpha_2)}{c\mu_1\mu_2(r_2\alpha_1-r_1\alpha_2)},$$

$$d=\frac{\gamma_2(r_2\mu_2\beta_1-r_1\mu_1\beta_2)}{\mu_1\mu_2^2c(r_2\alpha_1-r_1\alpha_2)},\quad N_1=\frac{r_2\beta_1\mu_2-r_1\beta_2\mu_1}{\mu_1\mu_2c(\alpha_1-\alpha_2)},$$

$$f=\frac{-m_1+\sqrt{m_1^2-4n_1}}{2},\quad h=\frac{-m_1-\sqrt{m_1^2-4n_1}}{2},$$

$$m_1=\frac{c(\mu_2\beta_1-\mu_1\beta_2)+\gamma_2\alpha_1\mu_1-\gamma_1\alpha_2\mu_2}{\mu_1\mu_2c(\alpha_1-\alpha_2)},\quad n_1=\frac{n}{\mu_1\mu_2},$$

and $a_1(0,\ 0)$ and $a_2(0,\ 0)$ are determined from (11) with allowance for the above comment.

The following expressions are obtained for the determination of the coefficients $a_k^2(Z,\ \mathrm{Fo})$ $(k = 1,\ 2)$:

$$a_1^2(Z,\ \mathrm{Fo})=\exp\left[\left(\frac{\beta_2}{A_2}-l_1\right)\xi\right]\exp\left[-\left(\frac{\beta_1}{A_1}+l_2\right)\eta\right]\times$$

$$\times\Big[\int_0^{l_1\xi}\mathrm{T}_1(\tau)J_0(2\sqrt{l_2\eta(\tau-l_1\xi)})\,d\tau+\int_0^{l_2\eta}\mathrm{T}_2(\tau)J_0(2\sqrt{l_1\xi(\tau-l_2\eta)})\,d\tau+$$

$$+a_1(0,\ 0)J_0(2\sqrt{-l_1l_2\xi\eta})\Big],$$

$$a_2^2(Z,\ \mathrm{Fo})=\exp\left[\left(\frac{\gamma_2}{c_1}-b_1\right)\xi\right]\exp\left[-\left(\frac{\gamma_1}{c_1}+b_2\right)\eta\right]\times$$

$$\times\Big[\int_0^{b_1\xi}\mathrm{X}_1(\tau)J_0(2\sqrt{b_2\eta(\tau-b_1\xi)})\,d\tau+$$

$$+\int_0^{b_2\eta}\mathrm{X}_2(\tau)J_0(2\sqrt{b_1\xi(\tau-b_1\eta)})\,d\tau+a_2(0,\ 0)J_0(2\sqrt{-b_1b_2\xi\eta})\Big],$$

where

$$\xi=Z-\mu_1\mathrm{Fo},\quad \eta=Z-\mu_2\mathrm{Fo},$$

TABLE 1. Results of a Calculation of the Average-Mass Temperature of Liquid in a Pipe with a Cross Section in the Form of an Equilateral Triangle by Eq. (12) (upper values) and from the Data of [6] (lower values)

Fo	Z			
	0,09	0,15	0,25	0,35
0,00	0,0003 0,0000	0,0003 0,0000	0,0003 0,0000	0,0003 0,0000
0,02	0,6498 0,5683	0,6498 0,5683	0,6498 0,5683	0,6498 0,5683
0,06	0,9250 0,9132	0,9497 0,9195	0,9497 0,9195	0,9497 0,9195
0,1	0,9270 0,9132	0,9841 0,9813	0,9920 0,9850	0,9920 0,9850
0,14	0,9270 0,9132	0,9842 0,9814	0,9987 0,9972	0,9987 0,9972
0,2	0,9270 0,9132	0,9842 0,9814	0,9988 0,9986	0,9999 0,9998
0,3	0,9270 0,9132	0,9842 0,9814	0,9988 0,9986	0,9999 0,9999

$$A_k = \alpha_k(\mu_1 - \mu_2),\ k = 1,\ 2,\ c_1 = c(\mu_1 - \mu_2),$$

$$l_1 = \frac{A_1}{A_1 - A_2}\left(\frac{\beta_2}{A_2} - \frac{\gamma_2}{c_1}\right),\ l_2 = \frac{A_2}{A_1 - A_2}\left(\frac{\gamma_1}{c_1} - \frac{\beta_1}{A_1}\right),$$

$$b_1 = \frac{l_1}{A_1}A_2,\ b_2 = -\frac{l_2}{A_2}A_1,$$

and $J_0(\varphi)$ is a zeroth-order Bessel function of the first kind.

The arbitrary functions $T_k(\tau)$ and $X_k(\tau)$, $k = 1, 2$, are determined from the conditions of continuity of the coefficient-functions $\alpha_k(Z, \text{Fo})$ on the characteristic curves

$$\begin{cases} a_1^2(0,\ \eta) = a_1^1(\text{Fo}), \\ a_1^2(\xi,\ 0) = a_1^3(Z), \\ a_2^2(0,\ \eta) = a_2^1(\text{Fo}), \\ a_2^2(\xi,\ 0) = a_2^3(Z). \end{cases}$$

The average-mass temperature of the liquid in the pipe is determined from the equation [6]

$$\tilde{\Theta}(Z,\ \text{Fo}) = \frac{\iint\limits_D \Theta(X,\ Y,\ Z,\ \text{Fo})\,W(X,\ Y)\,dD}{\iint\limits_D W(X,\ Y)\,dD}.$$

The results of a calculation of the average-mass temperature $\tilde{\Theta}(Z, \text{Fo})$ for different values of the angle β are presented in Fig. 1.

In order to test the correctness of the solution of the problem (5)-(8) obtained, we compared the results of the calculation of the average-mass temperature by Eq. (12) with $\beta = \pi/6$ with the results of a solution of the analogous problem for an equilateral triangle [6] (Table 1). The results of the comparison indicate the good agreement of the solution obtained with a known solution for a particular case of the problem.

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NONSTEADY HEAT TRANSFER FROM A CYLINDER WITH INJECTION

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The problem of nonsteady heat transfer from a cylinder in the presence of radial injection is analyzed. The heat flux at the surface of the cylinder is found from the assigned variation in the surface temperature by the method of [1].

The nonsteady heat transfer from a cylinder to an infinite external medium in the presence of radial injection is described by the problem

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\rho^2}-\frac{1-\mathrm{Pe}}{\rho}\cdot\frac{\partial}{\partial\rho}\right)T=0,$$
$$1\leqslant\rho<\infty,\quad 0<\tau<\infty, \tag{1}$$
$$T|_{\rho=1}=T_1(\tau);\quad T|_{\rho=\infty}=0;\quad T|_{\tau=0}=0,$$
$$\tau=R^2t/a;\quad \rho=r/R;\quad \mathrm{Pe}=uR/a.$$

The temperature gradient $q_1 = (\partial T/\partial\rho)_{\rho=1}$ at the surface of the cylinder is to be determined.

Despite the apparent simplicity of the problem, a solution by the operational method is very laborious, since the Laplace transform of the solution has the form

$$\bar{q}_1=\mathrm{Pe}/2+\sqrt{p}\,K'_{\mathrm{Pe}/2}(\sqrt{p})\,K^{-1}_{\mathrm{Pe}/2}(\sqrt{p})$$

(K is the MacDonald function and p is the parameter of the Laplace transform), and the primitive function is expressed through a complicated integral of special functions, even for the simplest case of Pe = 0 (see [2]).

Therefore, we carry out the solution following the method presented in [1], where a similar problem is analyzed for the equation

$$\left[\frac{\partial}{\partial\tau}-\alpha(\rho,\ \tau)\,\frac{\partial^2}{\partial\rho^2}-\beta(\rho,\ \tau)\,\frac{\partial}{\partial\rho}+\gamma(\rho,\ \tau)\right]T=0 \tag{2}$$

(the difference is only in the notation).

The solution of the problem (1) is expressed in the form of a series with respect to the derivative of the half-integral index of the assigned surface temperature $T_1(\tau)$:

$$-q_1(\tau)=\sum_{n=0}^{\infty}a_n(1)\,D^{\frac{1-n}{2}}\,T_1(\tau). \tag{3}$$

Here the operator for the fractional derivative of order ν is defined by the expression

$$D^{\nu}T_1(\tau)=\frac{1}{\Gamma(1-\nu)}\cdot\frac{d}{d\tau}\int_0^{\tau}(\tau-z)^{-\nu}\,T_1(z)\,dz,\quad \nu<1, \tag{4}$$

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